§ Covariant derivatives

Given a surface $S \subseteq \mathbb{R}^3$, recall that a vector field on S

$$X : S \longrightarrow \mathbb{R}^3 \qquad \text{(smooth)}$$

- is tangential if $X_p \in T_pS \quad \forall p \in S$
- is normal if $X_p \in (T_pS)^{\perp} \quad \forall p \in S$

Def$^{\underline{n}}$: $\mathfrak{X}(S) :=$ { tangential vector fields on S }

$\mathfrak{X}^{\perp}(S) :=$ { normal vector fields on S }

Q: How to differentiate vector fields in $\mathfrak{X}(S)$?

A: Covariant derivatives!

Def$^{\underline{n}}$: Given $X, Y \in \mathfrak{X}(S)$, define the covariant derivative of Y along X as

$$\boxed{\nabla_X Y := (D_X Y)^T}$$

where $(\cdot)^T$ refers to the tangential component of a vector based at $p \in S$ according to the orthogonal splitting

$\begin{pmatrix} \text{depends} \\ \text{on } p \end{pmatrix}$

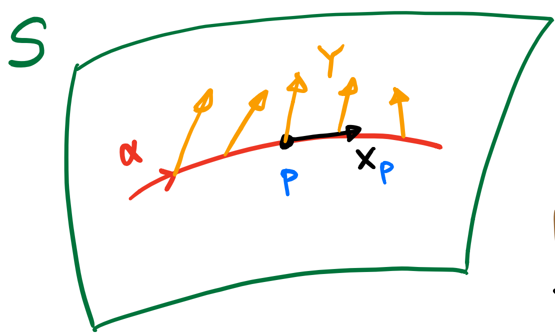
$$\boxed{\begin{array}{ccccc} (\mathbb{R}^3 =) T_p\mathbb{R}^3 & = & T_pS & \oplus & (T_pS)^{\perp} \\ \Downarrow & & \Downarrow & & \Downarrow \\ v & = & v^T & + & v^{\perp} \end{array}} \qquad (*)$$

<u>**Remarks**</u>: (1) Recall that $D_XY(p)$ depends <u>ONLY</u> on

(a) the vector X_p

and (b) the values of Y restricted to <u>ANY</u> curve

$\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ s.t. $\alpha(0) = p$, $\alpha'(0) = X_p$



$$D_XY(p) = \frac{d}{dt}\bigg|_{t=0} Y(\alpha(t))$$

Hence, this is well-defined even X, Y are only defined on S.

(2) $X, Y \in \mathfrak{X}(S) \Rightarrow \nabla_X Y \in \mathfrak{X}(S)$

We now study some important properties of ∇.

<u>Properties of ∇</u> : Let $X, Y, Z \in \mathfrak{X}(S)$, $f \in C^\infty(S)$,

a, b are real constants.

(1) Linearity in both variables:

$$\nabla_X (aY + bZ) = a\nabla_X Y + b\nabla_X Z$$

$$\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$$

(2) Liebniz rule: $\nabla_X (fY) = X(f)\, Y + f\nabla_X Y$

(3) Tensorial: $\nabla_{fX} Y = f\nabla_X Y$

(4) Torsion free: $\boxed{\nabla_X Y - \nabla_Y X = [X, Y]}$

(5) Metric compatibility:

$$\boxed{X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle}$$

<u>Remark:</u> The covariant derivative

$$\nabla : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$$

$$X \;,\; Y \longmapsto \nabla_X Y$$

is uniquely defined by properties (1) – (5) above!

"Fundamental Theorem of Riemannian geometry"

Proof: It follows from the fact that (1) - (5) are satisfied with "∇" replaced by "D" for vector fields in $\mathbb{R}^3$.

E.g. To prove (5),

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$$

$$(\text{Since } Y, Z \in \mathfrak{X}(S)) \quad = \langle \underbrace{(D_X Y)^T}_{\nabla_X Y}, Z\rangle + \langle Y, \underbrace{(D_X Z)^T}_{\nabla_X Z}\rangle$$

$\square$